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A Multiple Models Method for the Interval Uncertain Nonlinear System State Estimation

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Abstract: The following paper proposes a novel Multiple Models Method for observer design to solve the problem of state and parameter estimation of uncertain nonlinear time-varying parameters systems with unknown but bounded disturbance. Classically speaking, an interval observer is a special class of observers that generates a bounded interval vector for the real state vector in a guaranteed way under the assumption that the uncertainties are unknown but bounded; it gives an upper and lower estimate for the system states at each time instant (determining a certain interval for the estimated state variations). Several approaches have been developed and adapted to different kinds of models (linear, nonlinear, fuzzy, etc.). However, in the proposed approach, the objective is not to design an interval observer, but rather a classical Luemberger observer, based on an interval multiple model of the nonlinear system model. The novelty introduced in the paper is about proposing a new interval Multiple Models representation of the uncertain nonlinear system. The observer's gains are developed based on the Lyapunov stability theory proving that the state and parameter estimation errors are stable and converge to an origin-centred ball of a given radius to be minimized. The design conditions are formulated into linear matrix inequalities constraints, which can be efficiently solved. A numerical example is given to illustrate the design and validate the performance of the interval observers.

Keywords: Uncertain Systems, Sector Nonlinearity Approcha, Interval Multiple Models

1. Introduction

Nonlinear behavior appears in several engineering problems (mechanical, biological, biomedical, electrical, etc). Dealing with nonlinear systems has a wide and various aspect in control theory and goes from the modeling to the control, estimation and implementation. Apart of the nonlinear complexity, another difficulty for the system states design consists in the structural model errors or uncertainties; i.e. how modelling uncertainties is considered knowing that a major effect of these modelling errors is the cause of the mismatch between the model and the real behaviour of the system.

To overcome these difficulties, an alternative technique to the classical observer consists in dealing with the uncertainties and disturbances by determining certain upper and lower estimates for the system states at each time instant, which is known as set-membership or interval observers; i.e. set-based state estimators/observers started another branch of state estimation, where uncertainties are characterized by sets instead of random variables [1–3].

The aim of this paper is to develop a new method, combining the principle of set-membership interval observers with the multiple model approach. A first contribution, presented in a study is based on guaranteed bounds method [4]. It consists of an auxiliary dynamic system providing an upper estimation and a lower estimation for the solutions of the system considered under the assumption that the initial conditions and uncertain quantities are unknown but bounded. The mean of the interval can be considered as the point-wise estimate, whereas the interval width provides the admissible deviation from that value. The basic idea is to compute the set of admissible values for the state at each instant of time.

An Interval observer is a special class of observers that generates a bounded interval vector for the real state vector in a guaranteed way under the assumption that the uncertainties are unknown but bounded. Some of the basic concepts and the main developments in the designs and applications of interval observer for continuous-time, discrete-time (linear and non-linear), fuzzy and switched systems may be found in the following studies [5–10].

In the following paper, the considered approach is inspired from the interval observers or set-membership observers but quite different. Indeed, in the proposed approach, a step-by-step methodology to design an observer based on Multiple Model Structure to solve the problem of state and parameter estimation for uncertain nonlinear time-varying parameter systems will be given.

The idea is not to design an interval observer (with an upper and lower bound), but a classical Luemberger observer, based on an interval multiple model of the nonlinear system model.

It is important to highlight that in the following article, the resolution technique in order to deduce the observer gains is based on the Linear Matrix Inequality resolution. These LMIs constraints are deduced from a classical Lyapunov stability theory. One can find in the literature several contributions based on the LMI/Lyapunov development methods where the observation gain that guarantees both stability and positivity of the interval estimate errors is synthesized by solving a Linear Matrix Inequality (LMI) feasibility problem [14–17].

To illustrate the basic ideas, the development for the MM system with measurable premise variables is first detailed. The general case of nonlinear time-varying parameter systems where the parameters are inaccessible is then introduced. The stability conditions of the estimation errors are established and the reachable regions of convergence are characterized and optimized. The observer gains are derived by solving an LMI optimization problem obtained from the Lyapunov theory.

2. General Multiple Models Form

2.1. Multiple Models Representation of Nonlinear Systems

Consider a dynamic time-varying nonlinear system described by the following state equations:

$$\begin{cases} \dot{x}(t) = f(t, x, u, \theta) \\ y(t) = g(t, x, u, \theta) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the input, $y(t) \in \mathbb{R}^m$ is the system output and $\theta(t) \in \mathbb{R}^n$ a time-varying parameter vector. $f(.)$ and $g(.)$ are vector functions of dimension $\mathbb{R}^{n_x}$ and $\mathbb{R}^m$ respectively.

The nonlinear dynamic systems is then expressend into a convex combination of linear submodels:

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))(\mathcal{A}_{ij}x(t) + \mathcal{B}_{ij}u(t)) \\ y(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))(\mathcal{C}_{ij}x(t) + \mathcal{D}_{ij}u(t)) \end{cases} \tag{2}$$

with $X_i(.)$, for $X \in \{A, B, C, D\}$, are matrices functions respectively of dimensions $\mathbb{R}^{n_x \times n_x}$, $\mathbb{R}^{n_x \times n_u}$, $\mathbb{R}^{m \times n_x}$ and $\mathbb{R}^{m \times n_u}$. The functions $\mu_i(t)$ represent the weights of the submodels $(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_i)$ in the global model and satisfy the convex sum property:

$$\sum_{i=1}^{r}\mu_i(t) = 1, \qquad \sum_{i=1}^{r}\widetilde{\mu}_i(t) = 1 \forall t \tag{3}$$

Note that if the weighting functions $\mu_i(t)$ and $\widetilde{\mu}_i(t)$ are state or time-varying parameter dependent, they are called weighting functions with unmeasurable premise variables; in the same way, if they are input or output dependent (or the state and the parameters are known) they are referred as weighting functions with measurable premise variables.

2.2. Interval Multiple Models Representation

The unknown system states estimation is one of the most challenging and fundamental problem in many engineering fields, where the model uncertainty represents an additional difficulty to the model complexity. To overcome these kinds of problems and estimate the unknown states in the presence of large and fluctuating disturbances (parameter and/or model uncertainties), a new class of estimators have been developed recently known as set-membership state estimators and interval observers [11].

Interval observer is a class of observers that are used to evaluate the actual state of the dynamic process, such techniques are mainly based on guaranteed bounds method. It consists of an auxiliary dynamic system providing an upper estimation and a lower estimation for solutions of the system under consideration using the assumptions that the initial states conditions and uncertain quantities are unknown with known bounds. Such an approach can be used to deal with significant disturbances and provide component-wise information on possible solutions [11].

In the present contribution, based on [4], the uncertainty are modeled by means of interval parameters. Uncertainties affecting the parameters of the MM are taken into account by considering the lower and upper bounds of the matrices $[\mathcal{A}_{ij}]$ where the interval matrices $[\mathcal{A}_{ij}]$ characterize the $(i, j)^{th}$ local model.

For a linear output, the Interval Multiple Model (*IMM*) representation of a nonlinear system (1) is given by:

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))([\mathcal{A}_{ij}]x(t) + \mathcal{B}_{ij}u(t)) \\ y(t) &= Cx(t) \end{cases} \tag{4}$$

The lower and upper bounds of the matrices $[\mathcal{A}_{ij}]$ are respectively defined by:

$$\begin{cases} \underline{\mathcal{A}}_{ij} = \underline{A}_i + \sum_{k=1}^{n} \underline{\theta}_k^{\sigma_j^k} A_{ik} \\ \overline{\mathcal{A}}_{ij} = \overline{A}_i + \sum_{k=1}^{n} \overline{\theta}_k^{\sigma_j^k} A_{ik} \end{cases} \tag{5}$$

where:

$$\underline{A}_i = \begin{pmatrix} \underline{a}_{11}^i & \cdots & \underline{a}_{1n_x}^i \\ \underline{a}_{21}^i & \ddots & \\ \underline{a}_{n_x1}^i & \cdots & \underline{a}_{n_xn_x}^i \end{pmatrix}, \ \overline{A}_i = \begin{pmatrix} \overline{a}_{11}^i & \cdots & \overline{a}_{1n_x}^i \\ \overline{a}_{21}^i & \ddots & \\ \overline{a}_{n_x1}^i & \cdots & \overline{a}_{n_xn_x}^i \end{pmatrix} \tag{6}$$

and A_{ij} is the matrix that characterizes the influence of the parameter θ_j on the submodel i.

Since the bounds $\underline{a}_{lm}^i$, $\overline{a}_{lm}^i$, $\underline{\theta}_j$ and $\overline{\theta}_j$ are assumed to be known, the matrices $[\mathcal{A}_{ij}]$ are written as:

$$[\mathcal{A}_{ij}] = \mathcal{A}_{ij}^n + \sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{lm}^i]e_m^T + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{\theta,lm}^{ijk}]e_m^T \tag{7}$$

where $\mathcal{A}_{ij}^n = A_{in} + \sum_{k=1}^{n} \theta_k^{\sigma_j^k} A_{i^k}$, A_{in} and $\theta_k^{\sigma_j^k} A_{ik}$ represent the nominal system matrices where no uncertainties are taken into account. e_i refers to the vector where the element of coordinate i is equal to 1 and 0 elsewhere. The terms $[f_{lm}^i]$ and $[f_{\theta,lm}^{ij^k}]$ represent respectively the input, output or modeling and parameters uncertainties in terms of interval width, such that:

$$\begin{cases} |[f_{lm}^i]| \leq \delta_{lm}^i,\ \delta_{lm}^i = \overline{a}_{lm}^i - \underline{a}_{lm}^i \\ |[f_{\theta,lm}^{ijk}]| \leq \delta_{\theta,lm}^{ijk},\ \delta_{\theta,lm}^{ijk} = (\overline{\theta}_{\theta,lm})_k^{\sigma_j^k} A_{ik} - (\underline{\theta}_{\theta,lm})_k^{\sigma_j^k} A_{ik} \end{cases} \tag{8}$$

3. Observer Design

3.1. Measurable Premise Variables

Consider first the case of measurable (or known) premise variables. Let us define the following Luemberger observer for the *IMM* system (4):

$$\begin{cases} \dot{\hat{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))(\mathcal{A}_{ij}^n\hat{x}(t) + \mathcal{B}_{ij}u(t) + L_{ij}(y(t) - \hat{y}(t))) \\ \hat{y}(t) &= C\hat{x}(t) \end{cases} \tag{9}$$

The objective is to compute gains $L_{ij} \in \mathbb{R}^{n_x \times m}$ such that the state estimation error $e_x(t) = x(t) - \hat{x}(t)$ is asymptotically stable and to characterize the error convergence domain.

To this end, based on the Lyapunov theory, LMI conditions are derived to compute the gains L_{ij} in order that the error be quadratically stable and remains into a predefined set.

From (4), (7) and (9), the state estimation error dynamics can be calculated as:

$$\dot{e}_x(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))((\mathcal{A}_{ij}^n - L_{ij}C)e_x(t) + \left(\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{lm}^i]e_m^T + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{\theta,lm}^{ijk}]e_m^T\right) x(t)) \tag{10}$$

Considering a quadratic Lyapunov function $V(t) = e_x^T(t)Pe_x(t)$, the error is asymptotically stable if there exists a symmetric positive matrix $P = P^T > 0$ and matrices L_{ij}, $i = 1, \dots, r$, $j = 1, \dots, 2^n$ such that:

$$\begin{array}{l} \dot{V}(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\xi)\widetilde{\mu}_j(\theta)\left(e_x^T(t)\mathbb{S}(((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T))e_x(t)\right. \\ \left. +\mathbb{S}\left(\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_x^T(t)Pe_l\left([f_{lm}^i] + \sum_{k=1}^{n}[f_{\theta,lm}^{ijk}]\right)e_m^T x(t)\right)\right) < 0 \end{array} \tag{11}$$

with $\mathbb{S}(M) = M + M^T$ and $R_{ij} = PL_{ij}$.

In the proof of proposition 1, it will be shown that $\dot{V}(t) \leq -\varepsilon||e_x||_2^2 + \gamma$. A necessary stability condition is then given in the next proposition:

Proposition 1: The stability condition $\dot{V}(t) < 0$ (11) is satisfied if

$$\left\{ \begin{array}{c} Z_{ij} < 0 \\ and \\ \| e_x \|_2^2 > \frac{\gamma}{\varepsilon} \end{array} \right. \tag{12}$$

with

$$Z_{ij} = \mathbb{S}((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T) + PE\Omega_i^{-1}E^T P + \sum_{k=1}^{n} PE(\Omega_{i\theta}^{jk})^{-1}E^T P \tag{13}$$

$$\varepsilon = \min_{i=1:r,\, j=1:2^n} \lambda_{min}(-Z_{ij}) > 0 \tag{14}$$

and

$$\gamma = \max_{i=1:r,j=1:2^n} ||\overline{x}||_2^2\left(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n} E\Delta_{i\theta}^{jk}F\Omega_\theta^j F(\Delta_{i\theta}^{jk})^T E^T\right) \tag{15}$$

Proof Applying lemma 1:

Lemma 3.1. Consider two matrices X and Y with appropriate dimensions. For any positive scalars λ_1 and λ_2, the following property is verified:

$$-\lambda_1 X^T X + \lambda_1^{-1} Y^T Y \leq X^T Y + Y^T X \leq \lambda_2 X^T X + \lambda_2^{-1} Y^T Y \tag{16}$$

$\mathbb{S}\left(\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_x^T(t)Pe_l\left([f_{lm}^i] + \sum_{k=1}^{n}[f_{\theta,lm}^{ijk}]\right)e_m^T x(t)\right)$ can be bounded by:

$$\begin{array}{l} \mathbb{S}\left(\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_x^T(t)Pe_l\left([f_{lm}^i] + \sum_{k=1}^{n}[f_{\theta,lm}^{ij}]\right)e_m^T x(t)\right) \leq \\ \sum_{l=1}^{n_x}\sum_{m=1}^{n_x} \lambda_{lm}^i(\delta_{lm}^i)^2 x^T(t)e_m e_m^T x(t) + \sum_{l=1}^{n_x}\sum_{m=1}^{n_x}(\lambda_{lm}^i)^{-1} e_x^T(t)Pe_l e_l^T Pe_x(t) \\ +\sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} \lambda_{\theta,lm}^{ijk}(\delta_{\theta,lm}^{ijk})^2 x^T(t)e_m e_m^T x(t) + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x}(\lambda_{\theta,lm}^{ijk})^{-1} e_x^T(t)Pe_l e_l^T Pe_x(t) \end{array} \tag{17}$$

In order to simplify the notation and transform the double sum in matrices product, define:

$$\left\{ \begin{array}{l} \Delta_i = \text{diag}(\delta_{11}^i \dots \delta_{n_x 1}^i \delta_{12}^i \dots \delta_{n_x 2}^i \dots \delta_{1n_x}^i \dots \delta_{n_x n_x}^i) \\ \Delta_{i\theta}^{jk} = \text{diag}(\delta_{\theta,11}^{ijk} \dots \delta_{\theta,n_x 1}^{ijk}\delta_{\theta,12}^{ijk} \dots \delta_{\theta,n_x 2}^{ijk} \dots \delta_{\theta,1n_x}^{ijk} \dots \delta_{\theta,n_x n_x}^{ijk}) \\ \Omega_i = \text{diag}(\lambda_{11}^i \dots \lambda_{1n_x}^i \lambda_{12}^i \dots \lambda_{n_x 2}^i \dots \lambda_{n_x 1}^i \dots \lambda_{n_x n_x}^i) \\ \Omega_{\theta,i}^{jk} = \text{diag}(\lambda_{\theta,11}^{ijk} \dots \lambda_{\theta,1n_x}^{ijk}\lambda_{\theta,12}^{ijk} \dots \lambda_{\theta,n_x 2}^{ijk} \dots \lambda_{\theta,n_x 1}^{ijk} \dots \lambda_{\theta,n_x n_x}^{ijk}) \\ E = (I_{n_x} \dots I_{n_x}), E \in \mathbb{R}^{n_x \times n_x^2} \end{array} \right. \tag{18}$$

where $\text{diag}(M_1,\ldots,M_n)$ refers to a block diagonal matrix with the elements $M_1,\ldots,M_n$ on its diagonal.

The permutation matrix F is also introduced, such that:

$$F\Omega_i F = \text{diag}(\lambda_{11}^i \ldots \lambda_{n_x1}^i \lambda_{12}^i \ldots \lambda_{n_x2}^i \ldots \lambda_{1n_x}^i \ldots \lambda_{n_xn_x}^i) \tag{19}$$

The same transformation is applied to Ω_θ^j.

The stability condition $\dot{V}(t) < 0$ can be expressed as the following inequality:

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))(e_x^T(t)(\mathbb{S}((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T) + PE\Omega^{-1}E^T P\\ &+\sum_{k=1}^{n}PE(\Omega_{\theta,i}^{jk})^{-1}E^T P)e_x(t) + x^T(t)(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n}E\Delta_{i\theta}^{j}F\Omega_{\theta,i}^{jk}F(\Delta_{i\theta}^{jk})^T E^T)x(t)) < 0\end{aligned} \tag{20}$$

Since $x(t)$ is bounded $x(t) \in [\underline{x}, \overline{x}]$, then $x^T(t)(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n}E\Delta_{i\theta}^{jk}\ F\Omega_{i\theta}^{jk}F(\Delta_{i\theta}^{jk})^T E^T)x(t)$ can be bounded by $||\overline{x}||_2^2(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n} E\ \Delta_{i\theta}^{jk}F\ \Omega_{i\theta}^{jk}F(\Delta_{i\theta}^{jk})^T E^T)$.

Condition (20) is now written:

$$\sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))(e_x^T(t)Z_{ij}e_x(t) + ||\overline{x}||_2^2(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n}E\Delta_{i\theta}^{jk}F\Omega_{i\theta}^{jk}F(\Delta_{i\theta}^{jk})^T E^T)) < 0 \tag{21}$$

with

$$Z_{ij} = \mathbb{S}((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T) + PE\Omega_i^{-1}E^T P + \sum_{k=1}^{n}PE(\Omega_{i\theta}^{jk})^{-1}E^T P \tag{22}$$

Let us define

$$\varepsilon = \min_{i=1:r,\, j=1:2^n} \lambda_{min}(-Z_{ij}) \tag{23}$$

$$\gamma = \max_{i=1:r, j=1:2^n} ||\overline{x}||_2^2 \left(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n}E\Delta_{i\theta}^{jk}F\Omega_\theta^j F(\Delta_{i\theta}^{jk})^T E^T\right) \tag{24}$$

According to Lyapunov stability theory, $\dot{V}(t) < -\varepsilon \parallel e_x \parallel_2^2 + \gamma$. It follows that $\dot{V}(t) < 0$ if

$$\left\{\begin{array}{c} Z_{ij} < 0 \\ and \\ \parallel e_x \parallel_2^2 > \frac{\gamma}{\varepsilon}\end{array}\right. \tag{25}$$

which means that e_x is uniformly bounded and converges to a small origin-centered ball of radius $\sqrt{\frac{\gamma}{\varepsilon}}$.

Proposition 2: The state estimation error e_x is stable and converges to an origin-centered ball of radius $\sqrt{\frac{\gamma}{\varepsilon}}$ bounded by β if there exists $P = P^T > 0,\ R_{ij},\ \Omega,\ \Omega_\theta^k > 0$ solutions of the following optimization problem

$$\min_{i=1,\ldots,r,\, j=1,\ldots,2^n} \beta \tag{26}$$

s.t.

$$\begin{pmatrix} Z_{ij} & I \\ I & -\beta I \end{pmatrix} < 0 \tag{27}$$

with:

$$Z_{ij} = \sum_{k=1}^{n}\begin{pmatrix} (\mathcal{A}_{ij}^n)^T P + P\mathcal{A}_{ij}^n - C^T R_{ij}^T - R_{ij}C & PE & PE \\ * & -\Omega_i & 0 \\ * & * & -\Omega_{i\theta}^{jk}\end{pmatrix} \tag{28}$$

and

$$||\overline{x}||_2^2 \left(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n} E\Delta_{i\theta}^{jk} F\Omega_{i\theta}^{jk} F(\Delta_{i\theta}^{jk})^T E^T \right) < \beta \tag{29}$$

for $i = 1, \ldots, r,\ j = 1, \ldots, 2^n$.

The observer gains (9) are then given by $L_{ij} = P^{-1}R_{ij}$, $i = 1, \ldots, r$, $j = 1, \ldots, 2^n$.

Proof Applying a Schur's complement, $Z_{ij} < 0$ is equivalent to solve the following LMIs:

$$\sum_{k=1}^{n} \begin{pmatrix} (\mathcal{A}_{ij}^n)^T P + P\mathcal{A}_{ij}^n - C^T R_{ij}^T - R_{ij}C & PE & PE \\ * & -\Omega_i & 0 \\ * & * & -\Omega_{i\theta}^{jk} \end{pmatrix} < 0 \tag{30}$$

for $i = 1, \ldots, r,\ j = 1, \ldots, 2^n$. The observer gain are then given by $L_{ij} = P^{-1}R_{ij}$.

The objective is now to minimize the radius $\sqrt{\frac{\gamma}{\varepsilon}}$.

Let us consider a positive scalar β, such that:

$$||\overline{x}||_2^2 \left(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n} E\Delta_{i\theta}^{jk} F\Omega_{i\theta}^{jk} F(\Delta_{i\theta}^{jk})^T E^T \right) < \beta \tag{31}$$

$$\begin{pmatrix} Z_{ij} & I \\ I & -\beta I \end{pmatrix} < 0 \ for\ i = 1, \ldots, r,\ j = 1, \ldots, 2^n \tag{32}$$

From (32) we get:

$$(1/\beta)\, I < -Z_{ij},\ i = 1, \ldots, r,\ j = 1, \ldots, 2^n \tag{33}$$

implying that all the eigenvalues of $(-Z_{ij})$ are larger that $1/\beta$. As a consequence $1/\beta < \varepsilon$ holds.

Then, if $\gamma < \beta$ and $\varepsilon > 1/\beta$, it implies that the radius $\sqrt{\frac{\gamma}{\varepsilon}}$ is also bounded by β, so minimizing β implies to minimize the radius of the convergence origin-centered ball, which ends the proof.

3.2. Unmeasurable Premise Variables

As mentioned in the previous section, although the polytopic transformation may leads to MM with unmeasurable premise variables, most of the works on MM systems are devoted to models with known premise variables. In this section, we consider the case of unmeasurable premise variables. As for the measurable (known) premise variables case, an *IMM* is first proposed. The problem that appears here, in addition to the uncertainties, is the joint state and parameter estimation, since the premise variables are unmeasurable and depend on the state and parameters that need to be estimated.

Let us define the following joint state and parameter observer for the considered *ITS* system (4):

$$\begin{cases} \dot{\hat{x}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(\mathcal{A}_{ij}^n \hat{x}(t) + \mathcal{B}_{ij}u(t) + L_{ij}(y(t) - \hat{y}(t))) \\ \dot{\hat{\theta}}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n} \mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(K_{ij}(y(t) - \hat{y}(t)) - \alpha_{ij}\hat{\theta}(t)) \\ \hat{y}(t) &= C\hat{x}(t) \end{cases} \tag{34}$$

The objective is to find gains $L_{ij} \in \mathbb{R}^{n_x \times m}$, $K_{ij} \in \mathbb{R}^{n \times m}$ and $\alpha_{ij} \in \mathbb{R}^{n \times n}$ such that the estimation errors $e_x(t) = x(t) - \hat{x}(t)$ and $e_\theta(t) = \theta(t) - \hat{\theta}(t)$ are asymptotically stable and bounded. To ensure this objective, LMI conditions based on the Lyapunov theory are given.

The state estimation error dynamics cannot be easily computed directly since the premise variables are unmeasurable and depend on the state and parameters that need to be estimated. To overcome this difficulty, the state equation (4) is rewritten as follows:

$$\begin{aligned} \dot{x}(t) = &\sum_{i=1}^{r}\sum_{j=1}^{2^n} (\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))([\mathcal{A}_{ij}]x(t) + \mathcal{B}_{ij}u(t)) \\ &+(\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t)) - \mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t)))([\mathcal{A}_{ij}]x(t) + \mathcal{B}_{ij}u(t))) \end{aligned} \tag{35}$$

This form allows a better comparison of $x(t)$ with $\hat{x}(t)$ since $\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))$ not only appears in (34) but also in (35). Let us define:

$$\begin{aligned} \Delta A(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n}(\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t)) - \mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t)))[\mathcal{A}_{ij}] \\ &= (\mathcal{A}+\mathcal{F})\Sigma_A(t)E_A \end{aligned} \tag{36}$$

and

$$\begin{aligned} \Delta B(t) &= \sum_{i=1}^{r}\sum_{j=1}^{2^n}(\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t)) - \mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t)))\mathcal{B}_{ij} \\ &= \mathcal{B}\Sigma_B(t)E_B \end{aligned} \tag{37}$$

with

$$\begin{aligned} &\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{r2^n} \end{bmatrix}, \mathcal{F} = \begin{bmatrix} \mathcal{F}_{11} & \dots & \mathcal{F}_{r2^n} \end{bmatrix}, \\ &\mathcal{B} = \begin{bmatrix} \mathcal{B}_{11} & \dots & \mathcal{B}_{r2^n} \end{bmatrix} \\ &\Sigma(t) = \text{diag}(\gamma_{11}(t), \dots, \gamma_{r2^n}(t)), \\ &\gamma_{ij}(t) = \mu_i(x(t))\widetilde{\mu}_j(\theta(t)) - \mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{\theta}(t)) \\ &E_A = \begin{bmatrix} I_{n_x} & \dots & I_{n_x} \end{bmatrix}^T, \ E_B = \begin{bmatrix} I_{n_u} & \dots & I_{n_u} \end{bmatrix}^T \\ &\mathcal{F}_{ij} = \sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{lm}^i]e_m^T + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{\theta,lm}^{ijk}]e_m^T \end{aligned} \tag{38}$$

Thanks to the convex sum property, we also have

$$-1 \le \gamma_{ij}(t) \le 1 \tag{39}$$

which implies from definitions (38)

$$\Sigma_A^T(t)\Sigma_A(t) \le I, \quad \Sigma_B^T(t)\Sigma_B(t) \le I \tag{40}$$

Using (36) and (37), the system (35) is then written as:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(([\mathcal{A}_{ij}] + \Delta A(t))x(t) + (\mathcal{B}_{ij} + \Delta B(t))u(t)) \tag{41}$$

From equations (41), (7) and (34), the dynamics of the state estimation error is given by

$$\dot{e}_x(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))\left((\mathcal{A}_{ij}^n - L_{ij}C)e_x(t) + (\Delta A(t) + \Lambda)x(t) + \Delta B(t)u(t)\right) \tag{42}$$

with:

$$\Lambda = \sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{lm}^i]e_m^T + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x} e_l[f_{\theta,lm}^{ijk}]e_m^T$$

Define the parameter estimation error $e_\theta(t)$ as

$$e_\theta(t) = \theta(t) - \hat{\theta}(t) \tag{43}$$

From the observer definition (34), the dynamics of this error is given by

$$\dot{e}_\theta(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))\left(\dot{\theta}(t) - K_{ij}Ce_x(t) + \alpha_{ij}\theta(t) - \alpha_{ij}e_\theta(t)\right) \tag{44}$$

Let us first consider the state estimation error $e_x(t)$. Considering a quadratic Lyapunov function $V_x(t) = e_x^T(t)P_xe_x(t)$, the error is asymptotically stable if there exists a symmetric positive matrix $P_x = P_x^T > 0 \in \mathbb{R}^{n_x \times n_x}$ such that:

$$\begin{array}{l}\dot{V}_x(t) = \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\xi(t))\widetilde{\mu}_j(\theta(t))\left(e_x^T(t)\mathbb{S}(((\mathcal{A}_{ij}^n)^TP - C^TR_{ij}^T))e_x(t)\right.\\ +\mathbb{S}\left(x^T(t)(\Delta A^T(t)+\Lambda^T)P_xe_x(t)\right)+\mathbb{S}\left(u^T(t)\Delta B^T(t)P_xe_x(t)\right)<0\end{array} \tag{45}$$

As in the previous section, it will be shown that $\dot{V}_x(t) \leq -\varepsilon_x||e_x||^2 + \gamma_x$. A necessary stability condition is then given in the next proposition.

Proposition 3: The stability condition $\dot{V}_x(t) < 0$ (45) is verified if

$$\left\{\begin{array}{c} Z_{ijx} < 0 \\ and \\ \parallel e_x \parallel_2^2 > \frac{\gamma_x}{\varepsilon_x}\end{array}\right. \tag{46}$$

with

$$\begin{array}{l}Z_{ijx} = \mathbb{S}((\mathcal{A}_{ij}^n)^TP - C^TR_{ij}^T) + PE\Omega^{-1}E^TP + \sum_{k=1}^{n}PE(\Omega_{\theta,i}^{jk})^{-1}E^TP\\ +\lambda_A^{-1}P(\mathcal{A}+\mathcal{F}_\delta)(\mathcal{A}+\mathcal{F}_\delta)^TP + \lambda_B^{-1}P\mathcal{B}\mathcal{B}^TP\end{array} \tag{47}$$

$$\varepsilon_x = \min_{i=1:r,\, j=1:2^n}\lambda_{min}(-Z_{ijx}) \tag{48}$$

and

$$\gamma_x = \max_{i=1:r,j=1:2^n}(||\overline{x}||_2^2(E\Delta_iF\Omega_iF\Delta_i^TE^T + \sum_{k=1}^{n}E\Delta_{\theta i}^{jk}F\Omega_{\theta i}^{jk}F(\Delta_{\theta i}^{jk})^TE^T + \lambda_AE_A^TE_A) + ||\overline{u}||_2^2\lambda_BE_B^TE_B) \tag{49}$$

Proof Applying lemma 3.1 and basing on the same reasoning as in the previous subsection ($x(t) \in [\underline{x}, \overline{x}]$, $u(t) \in [\underline{u}, \overline{u}]$), the following terms of the time derivative of the Lyapunov function (45) are bounded as follows:

1.
$$\begin{array}{l}\mathbb{S}\left(x^T(t)\Lambda^TP_xe_x(t)\right) \leq \sum_{l=1}^{n_x}\sum_{m=1}^{n_x}\lambda_{lm}^i(\delta_{lm}^i)^2x^T(t)e_me_m^Tx(t) + \sum_{l=1}^{n_x}\sum_{m=1}^{n_x}(\lambda_{lm}^i)^{-1}e_x^T(t)Pe_le_l^TPe_x(t)\\ +\sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x}\lambda_{\theta,lm}^{ijk}(\delta_{\theta,lm}^{ijk})^2x^T(t)e_me_m^Tx(t) + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x}(\lambda_{\theta,lm}^{ijk})^{-1}e_x^T(t)Pe_le_l^TPe_x(t)\end{array} \tag{50}$$

2. From (36) and lemma 3.1, $\mathbb{S}\left(x^T(t)\Delta A^T(t)P_xe_x(t)\right)$ can be bounded by:

$$\mathbb{S}\left(x^T(t)\Delta A^TP_xe_x(t)\right) \leq \lambda_Ax^T(t)E_A^T\Sigma_A^T(t)\Sigma_A(t)E_Ax(t) + \lambda_A^{-1}e_x^T(t)P(\mathcal{A}+\mathcal{F})(\mathcal{A}^\mathcal{T}+\mathcal{F}^\mathcal{T})Pe_x(t) \tag{51}$$

From the convex sum property, $\Sigma_A^T(t)\Sigma_A(t) \leq I$, which leads to:

$$\lambda_Ax^T(t)E_A^T\Sigma_A^T(t)\Sigma_A(t)E_Ax(t) \leq \lambda_Ax^T(t)E_A^TE_Ax(t) \tag{52}$$

The term $\lambda_A^{-1}e_x^T(t)P(\mathcal{A}+\mathcal{F})(\mathcal{A}^\mathcal{T}+\mathcal{F}^\mathcal{T})Pe_x(t)$ can also be bounded as:

$$\lambda_A^{-1}e_x^T(t)P(\mathcal{A}+\mathcal{F})(\mathcal{A}^\mathcal{T}+\mathcal{F}^\mathcal{T})Pe_x(t) \leq \lambda_A^{-1}e_x^T(t)P(\mathcal{A}+\mathcal{F}_\delta)(\mathcal{A}^\mathcal{T}+\mathcal{F}_\delta^\mathcal{T})Pe_x(t) \tag{53}$$

where $\mathcal{F}_\delta$ is defined as:

$$\mathcal{F}_\delta = \left[\begin{array}{ccc}(\mathcal{F}_{11})_\delta & \ldots & (\mathcal{F}_{r2^n})_\delta\end{array}\right] \tag{54}$$

with

$$(\mathcal{F}_{ij})_\delta = \sum_{l=1}^{n_x}\sum_{m=1}^{n_x}e_l\delta_{\theta,lm}^{ijk}e_m^T + \sum_{k=1}^{n}\sum_{l=1}^{n_x}\sum_{m=1}^{n_x}e_l\delta_{\theta,lm}^{ijk}e_m^T \tag{55}$$

3.
$$\mathbb{S}\left(u^T(t)\Delta B^T(t)P_x e_x(t)\right) \leq \lambda_B u^T(t)E_B^T(t)E_B u(t) + \lambda_B^{-1} e_x^T(t)P\mathcal{B}\mathcal{B}^T(t)Pe_x(t) \tag{56}$$

Considering definitions (18), the Lyapunov function derivative $\dot{V}_x(t)$ (45) is then bounded by:

$$\begin{aligned}\dot{V}_x(t) \leq & \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(e_x^T(t)(\mathbb{S}((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T) + PE\Omega^{-1}E^T P\sum_{k=1}^{n}PE(\Omega_{\theta,i}^{jk})^{-1}E^T P \\ & +\lambda_A^{-1}P(\mathcal{A}+\mathcal{F}_\delta)(\mathcal{A}+\mathcal{F}_\delta)^T P + \lambda_B^{-1}P\mathcal{B}\mathcal{B}^T P)e_x(t)) + x^T(t)(E\Delta_i F\Omega_i F\Delta_i^T E + \sum_{k=1}^{n}E\Delta_{\theta i}^{jk}F\Omega_{\theta i}^{jk}F(\Delta_{\theta i}^{jk})^T E \\ & +\lambda_A E_A^T E_A))x(t) + \lambda_B u^T(t)E_B^T E_B u(t)\end{aligned} \tag{57}$$

Since $x(t)$ and $u(t)$ are bounded as $x(t) \in [\underline{x}, \overline{x}]$, $u(t) \in [\underline{u}, \overline{u}]$, $\dot{V}_x(t)$ is bounded by:

$$\begin{aligned}\dot{V}_x(t) \leq & \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(e_x^T(t)Z_{ijx}e_x(t) + ||\overline{x}||_2^2(E\Delta_i F\Omega_i F\Delta_i^T E \\ & +\sum_{k=1}^{n}E\Delta_{\theta i}^{jk}F\Omega_{\theta i}^{jk}F(\Delta_{\theta i}^{jk})^T E + \lambda_A E_A^T E_A)) + ||\overline{u}||_2^2\lambda_B E_B^T E_B\end{aligned} \tag{58}$$

with

$$\begin{aligned}Z_{ijx} = & \mathbb{S}((\mathcal{A}_{ij}^n)^T P - C^T R_{ij}^T) + PE\Omega^{-1}E^T P + \sum_{k=1}^{n}PE(\Omega_{\theta,i}^{jk})^{-1}E^T P + \\ & \lambda_A^{-1}P(\mathcal{A}+\mathcal{F}_\delta)(\mathcal{A}+\mathcal{F}_\delta)^T P + \lambda_B^{-1}P\mathcal{B}\mathcal{B}^T P\end{aligned} \tag{59}$$

According to Lyapunov stability theory, e_x is uniformly bounded and converges to a small origin-centered ball of radius $\sqrt{\frac{\gamma_x}{\varepsilon_x}}$ bounded by β_x s.t.

$$(||\overline{x}||_2^2(E\Delta_i F\Omega_i F\Delta_i^T E^T + \sum_{k=1}^{n}E\Delta_{\theta i}^{jk}F\Omega_{\theta i}^{jk}F(\Delta_{\theta i}^{jk})^T E^T + \lambda_A E_A^T E_A) + ||\overline{u}||_2^2\lambda_B E_B^T E_B)_{i=1,\ldots,r,j=1,\ldots,2^n} < \beta_x \tag{60}$$

Second point, we consider the parameter estimation error $e_\theta(t)$. Considering a quadratic Lyapunov function $V_\theta(t) = e_\theta^T(t)P_\theta e_\theta(t)$, the error is asymptotically stable if there exists a symmetric positive matrix $P_\theta = P_\theta^T > 0 \in \mathbb{R}^{n\times n}$ such that:

$$\begin{aligned}\dot{V}_\theta(t) = & \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(-e_\theta^T(t)\mathbb{S}(R_{\theta ij}^2)e_\theta(t) + \mathbb{S}(e_\theta^T(t)P_\theta\dot{\theta}(t)) - \mathbb{S}(e_\theta^T(t)R_{\theta ij}^1 C\dot{e}_x(t)) \\ & +\mathbb{S}(e_\theta^T(t)R_{\theta ij}^2\theta(t))) < 0\end{aligned} \tag{61}$$

where $R_{\theta ij}^1 = P_\theta K_{ij}$ and $R_{\theta ij}^2 = P_\theta\alpha_{ij}$.

Proposition 4: The stability condition $\dot{V}_\theta(t) < 0$ (61) is verified if:

$$\begin{cases} Z_{ij\theta} < 0 \\ and \\ \parallel e_\theta \parallel_2^2 > \frac{\gamma_\theta}{\varepsilon_\theta} \end{cases} \tag{62}$$

with:

$$Z_{ij\theta} = -R_{\theta ij}^2 - (R_{\theta ij}^2)^T + \lambda_{1\theta}^{-1}P_\theta P_\theta + \lambda_{2\theta}^{-1}R_{\theta ij}^1(R_{\theta ij}^1)^T + \lambda_{3\theta}^{-1}R_{\theta ij}^2(R_{\theta ij}^2)^T \tag{63}$$

Let us define

$$\varepsilon_\theta = \min_{i=1:r,\, j=1:2^n}\lambda_{min}(-Z_{ij\theta}) \tag{64}$$

and

$$\gamma_\theta = \max_{i=1:r,j=1:2^n}(\lambda_{1\theta}||\overline{\dot{\theta}}||_2^2 + \lambda_{2\theta}C^T C\beta_x^2 + \lambda_{3\theta}||\overline{\theta}||_2^2) \tag{65}$$

Proof Applying lemma 3.1, $\dot{V}_\theta(t)$ can be bounded by:

$$\begin{aligned}\dot{V}_\theta(t) \leq \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(e_\theta^T(t)(R_{\theta ij}^2-(R_{\theta ij}^2)^T+\lambda_{1\theta}^{-1}P_\theta P_\theta+\lambda_{2\theta}^{-1}R_{\theta ij}^1(R_{\theta ij}^1)^T\\+\lambda_{3\theta}^{-1}R_{\theta ij}^2(R_{\theta ij}^2)^T)e_\theta(t)+\lambda_{1\theta}\dot{\theta}^T(t)\dot{\theta}(t)+\lambda_{2\theta}e_x^T(t)C^TCe_x(t)+\lambda_{3\theta}\theta^T(t)\theta(t)\end{aligned} \tag{66}$$

since $\theta(t)$, $e_x(t)$ and $\dot{\theta}(t)$ are bounded $\theta(t)\in[\underline{\theta},\overline{\theta}]$, $e_x(t)\in[-\beta_x,\beta_x]$ and $\dot{\theta}(t)\in[\underline{\dot{\theta}},\overline{\dot{\theta}}]$. The inequality (66) is bounded by:

$$\dot{V}_\theta(t) \leq \sum_{i=1}^{r}\sum_{j=1}^{2^n}\mu_i(\hat{\xi}(t))\widetilde{\mu}_j(\hat{\theta}(t))(e_\theta^T(t)Z_{ij\theta}e_\theta(t)+\lambda_{1\theta}||\overline{\dot{\theta}}||_2^2+\lambda_{2\theta}C^TC\beta_x^2+\lambda_{3\theta}||\overline{\theta}||_2^2 \tag{67}$$

According to Lyapunov stability theory, $\dot{V}_\theta(t) < -\varepsilon_\theta \parallel e_\theta \parallel_2^2 +\gamma_\theta$, which means that $e_\theta(t)$ is uniformly bounded and converges to a small origin-centered ball of radius $\sqrt{\frac{\gamma_\theta}{\varepsilon_\theta}}$ bounded by β_θ s.t.

$$((\lambda_{1\theta}||\overline{\dot{\theta}}||_2^2+\lambda_{2\theta}C^TC\beta_x^2+\lambda_{3\theta}||\overline{\theta}||_2^2))_{i=1,\dots,r,j=1,\dots,2^n}<\beta_\theta \tag{68}$$

To sum up, the state and parameter estimation errors are stable and converge to an origin-centered ball of radius $\sqrt{\frac{\gamma_x}{\varepsilon_x}}$ and $\sqrt{\frac{\gamma_\theta}{\varepsilon_\theta}}$ respectively bounded by β_x and β_θ if there exists $P_x=P_x^T>0,\ P_\theta=P_\theta^T>0,\ R_{ij},\ R_{\theta ij}^1,\ R_{\theta ij}^2\ \Omega,\ \Omega_\theta^k>0$, $\lambda_A>0$, $\lambda_B>0$, $\lambda_{1\theta}>0$, $\lambda_{2\theta}>0$ and $\lambda_{3\theta}$ solutions of the following optimization problems

$$\begin{array}{c}\min_{i=1,\dots,r,\,j=1,\dots,2^n}\beta_x\\ \min_{i=1,\dots,r,\,j=1,\dots,2^n}\beta_\theta\end{array} \tag{69}$$

s.t.

$$\begin{pmatrix} Z_{ijx} & I\\ I & -\beta I\end{pmatrix}<0 \qquad and \qquad \begin{pmatrix} Z_{ij\theta} & I\\ I & -\beta I\end{pmatrix}<0 \tag{70}$$

with:

$$Z_{ijx}=\sum_{k=1}^{n}\begin{pmatrix} Z_{ij}^1 & PE & PE & P(\mathcal{A}+\mathcal{F}_\delta) & P\mathcal{B}\\ * & -\Omega_i & 0 & 0 & 0\\ * & * & -\Omega_{i\theta}^{jk} & 0 & 0\\ * & * & * & -\lambda_A & 0\\ * & * & * & * & -\lambda_B\end{pmatrix} \tag{71}$$

with $Z_{ij}^1=\mathbb{S}((\mathcal{A}_{ij}^n)^TP-C^TR_{ij}^T)$

$$Z_{ij\theta}=\begin{pmatrix} -R_{\theta ij}^2-(R_{\theta ij}^2)^T & P_\theta & R_{\theta ij}^1 & R_{\theta ij}^2\\ * & -\lambda_{1\theta} & 0 & 0\\ * & * & -\lambda_{2\theta} & -\lambda_{3\theta}\end{pmatrix} \tag{72}$$

and

$$\begin{cases}(||\overline{x}||_2^2(E\Delta_iF\Omega_iF\Delta_i^TE^T+\sum_{k=1}^{n}E\Delta_{\theta i}^{jk}F\Omega_{\theta i}^{jk}F(\Delta_{\theta i}^{jk})^TE^T+\lambda_AE_A^TE_A)+||\overline{u}||_2^2\lambda_BE_B^TE_B)_{i=1,\dots,r,j=1,\dots,2^n}<\beta_x\\ ((\lambda_{1\theta}||\overline{\dot{\theta}}||_2^2+\lambda_{2\theta}C^TC\beta_x^2+\lambda_{3\theta}||\overline{\theta}||_2^2))_{i=1,\dots,r,j=1,\dots,2^n}<\beta_\theta\end{cases} \tag{73}$$

The observer gains (34) are then given by $L_{ij}=P^{-1}R_{ij}$, $K_{ij}=P_\theta^{-1}R_{\theta ij}^1$ and $\alpha_{ij}=P_\theta^{-1}R_{\theta ij}^2$.

4. Example

To illustrate the proposed methodology, let us consider the following system with additive disturbances described by (74)

$$\begin{cases}\dot{x}(t) &= A(\theta(t))x(t)+d(t)\\ y(t) &= Cx(t)\end{cases} \tag{74}$$

where $x \in \mathbb{R}^2$ is the state and

$$A(\theta(t)) = \begin{pmatrix} -0.632 - 0.8\sin(t) & 0.5\cos(3t) \\ -0.7\cos(2t) & 0.3\sin(t) \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{75}$$

The input $d(t)$ is defined by $d(t) = \begin{pmatrix} 0.1 + 0.2\sin(0.5t) & 0.1 + 0.5\cos(1.5t) \end{pmatrix}^T$.

First step, let us write the system equations (74) in a MM form. The state equation can be re-written as:

$$\dot{x}(t) = A(\theta(t))x(t) + \frac{d(t)}{x_1(t)}x_1(t) \tag{76}$$

Equation (76) is decomposed as the following:

$$\dot{x}(t) = \begin{pmatrix} a_{11} + \rho_{11}\theta_1(t) + \xi_1(t) & \rho_{12}\theta_2(t) \\ \rho_{21}\theta_3(t) + \xi_2(t) & \rho_{22}\theta_1(t) \end{pmatrix} x(t) \tag{77}$$

with

$$a_{11} = -0.632,\ \theta_1(t) = \sin(t),\ \theta_2(t) = \cos(3t),\ \theta_3(t) = \cos(2t)$$

$$\xi_1(t) = \tfrac{d_1(t)}{x_1(t)},\ \xi_2(t) = \tfrac{d_2(t)}{x_1(t)}$$

$$\rho_{11} = -0.8,\ \rho_{12} = 0.5,\ \rho_{21} = -0.7,\ \rho_{22} = 0.3$$

For the nonlinearities $\xi_1(t)$, $\xi_2(t)$ and the time-varying parameters $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$, by applying the above MM transformation (section 2.1), the nonlinear system (74) is written as:

$$\dot{x}(t) = \sum_{i=1}^{4}\sum_{j=1}^{8} \mu_i(x_1(t))\widetilde{\mu}_j(\theta(t))\mathcal{A}_{ij}x(t) \tag{78}$$

with for $i = 1, \ldots, 4$:

$$\begin{cases} \mu_i^1(x_1(t)) = \dfrac{\xi_i(t) - \xi_i^2}{\xi_i^1 - \xi_i^2},\ x_1(t) \neq 0 \\ \mu_i^1(x_1(t)) = 1,\ x_1(t) = 0 \\ \mu_i^2(x_1(t)) = \dfrac{\xi_i^1 - \xi_i(t)}{\xi_i^1 - \xi_i^2},\ x_1(t) \neq 0 \\ \mu_i^2(x_1(t)) = 0,\ x_1(t) = 0 \\ \xi_i^1 = \max \xi_i(t) \\ \xi_i^2 = \min \xi_i(t) \end{cases} \tag{79}$$

and for $j = 1, \ldots, 8$ (each submodel j is defined for a triplet $(\sigma_j^1, \sigma_j^2, \sigma_j^3)$):

$$\begin{cases} \widetilde{\mu}_j(\theta(t)) = \widetilde{\mu}_1^{\sigma_j^1}(\theta_1(t))\widetilde{\mu}_2^{\sigma_j^2}(\theta_2(t))\widetilde{\mu}_3^{\sigma_j^3}(\theta_3(t)) \\ \widetilde{\mu}_k^1(\theta_k(t)) = \dfrac{\theta_k(t) - \theta_k^2}{\theta_k^1 - \theta_k^2} \\ \widetilde{\mu}_k^2(\theta_k(t)) = \dfrac{\theta_k^1 - \theta_k(t)}{\theta_k^1 - \theta_k^2} \\ \mathcal{A}_{ij} = A_i + \theta_1^{\sigma_j^1}A_{i1} + \theta_2^{\sigma_j^2}A_{i2} + \theta_3^{\sigma_j^3}A_{i3} \\ \theta_k^1 = \max \theta_k(t) \\ \theta_k^2 = \min \theta_k(t) \end{cases} \tag{80}$$

where σ_j^k is equal to 1 or 2 and indicates which partition of the k^{th} parameter $k = 1, 2, 3$ ($\widetilde{\mu_k}^1$ or $\widetilde{\mu_k}^2$) is involved in the j^{th} submodel. The relation between the submodel number j and the σ_j^k index is given by the following equation

$$j = 2^2\sigma_j^1 + 2^1\sigma_j^2 + 2^0\sigma_j^3 - (2^1 + 2^2) \tag{81}$$

The matrices $\mathcal{A}_{ij}$ are defined as follows:

$$\begin{cases} \mathcal{A}_{ij} = A_i + \theta_1^{\sigma_j^1} A_{i1} + \theta_2^{\sigma_j^2} A_{i2} + \theta_3^{\sigma_j^3} A_{i3} \\ A_1 = \begin{pmatrix} a_{11} + \xi_1^1 & 0 \\ \xi_2^1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} a_{11} + \xi_1^1 & 0 \\ \xi_2^2 & 0 \end{pmatrix} \\ A_3 = \begin{pmatrix} a_{11} + \xi_1^2 & 0 \\ \xi_2^1 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} a_{11} + \xi_1^2 & 0 \\ \xi_2^2 & 0 \end{pmatrix} \\ A_{i1} = \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{22} \end{pmatrix}, A_{i2} = \begin{pmatrix} 0 & \rho_{12} \\ 0 & 0 \end{pmatrix}, \\ A_{i3} = \begin{pmatrix} 0 & 0 \\ \rho_{21} & 0 \end{pmatrix} \end{cases} \tag{82}$$

It is assumed that the exogenous input $d(t)$ is subject to uncertainties but remains bounded such that:

$$\begin{pmatrix} d_{1m}(t) \\ d_{2m}(t) \end{pmatrix} \leq d(t) = \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} \leq \begin{pmatrix} d_{1M}(t) \\ d_{2M}(t) \end{pmatrix} \tag{83}$$

with:

$$\begin{cases} -0.5 & \leq d_{1m}(t) \leq & -0.1 \\ 0.3 & \leq d_{1M}(t) \leq & 0.7 \\ -1.1 & \leq d_{2m}(t) \leq & -0.1 \\ 0.1 & \leq d_{2M}(t) \leq & 1.1 \end{cases} \tag{84}$$

Our objective is to synthesis an observer $\hat{x}_1(t)$ for $x_1(t)$ (since the second state $x_2(t)$ is measured) by applying the above proposed approach. Solving the proposed optimization problem under LMIs constraints, a state observer with unmeasurable premise variable ($x_1(t)$) is designed.

In figure 1 are depicted respectively the system state $x_1(t)$ (subject to uncertainties) with its lower and upper width and its estimate $\hat{x}_1(t)$.

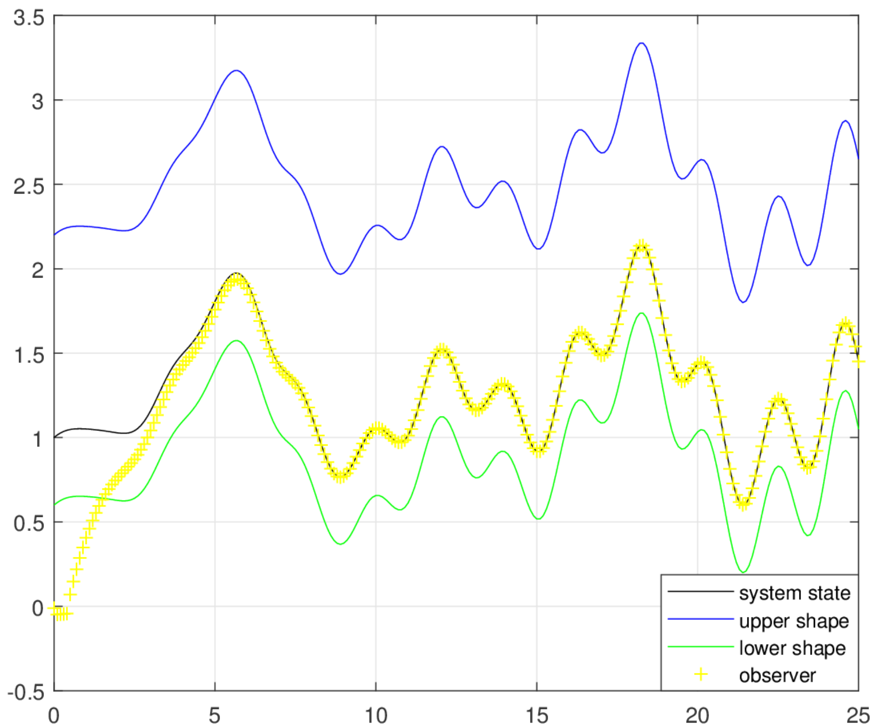

Figure 1. *System state $x_1(t)$ with its upper and lower width and its estimate $\hat{x_1}(t)$.*

As the figure 1 shows, the state $x_1(t)$ is well estimated with the proposed approach. The calculated gains are equal to:

$$L_1 = \begin{pmatrix} 3.81 \\ 73.01 \end{pmatrix}, L_2 = \begin{pmatrix} -1.83 \\ 73.36 \end{pmatrix}, L_3 = \begin{pmatrix} 3.78 \\ 71.72 \end{pmatrix}, L_4 = \begin{pmatrix} -1.52 \\ 71.72 \end{pmatrix} \tag{85}$$

Note that for computational reasons, the gains L_{ij}, $i = 1, \ldots, 4$ and $j = 1, \ldots, 8$ were set equal to L_i (meaning that $L_{i1} = L_{i2} = \cdots = L_{i8} = L_i$).

5. Conclusion

This work deals is devoted to a Multiple Models observer design for nonlinear time-varying parameters subject to uncertainties. The uncertain nonlinear system is represented in an *IMM* form and the observer is designed based on the nominal system. It is the first time that the observer design is treated in such a way. The proposed approach gives stability conditions for the estimation errors in terms of LMIs constraints and enables to characterize and optimize a reachable regions for both the states and time-varying parameters.

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